On principal eigenpair of temporal-joined adjacency matrix for spreading phenomenon

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Abstract. This paper reports a framework of analysing of spreading herbivore of individual-based system with time evolution network $\widetilde{A}(t)$. By employing a sign function $\theta_1(x)$, $\theta_1(0) = 0$, $\theta_1(x) = 1$ $x \in \mathbb{N}$, the dynamic equation of spreading is in a matrix multiplication expression. Based on that, a method of combining temporal network is reported. The risk of been-spread and the ability-to-spread can be illustrated by the principal eigenpair of temporal-joined matrix in a system. The principal eigenpair of post-joined matrix can estimate the step number to the furtherest agent S_i in a non-time evolution network system $\widetilde{A}(t) = \widetilde{A}$ as well.

Keywords: Agent model · epidemic · contact network

1 Introduction

Various applications of network crossing social and natural disciplinaries [1]. As a mathematical abstraction method, network describes the interactions among elements inside of system as links among nodes. The non-direction and static network gets the highest abstraction level and gives us uncountable results of explaining the dynamic properties of system. Temporal network [2] reduced the level of abstraction to contain essential dynamic information of system. Spectral method and eigensystem decomposition of network adjacency matrix, or of Laplacian matrix, are fortherword abstraction method, which reveals the topology properties of network [3] and dynamic properties of system [3, 4]. But we do not have a good framework to combine spectral method and temporal network nowadays. Spreading on network is a scientific problem that wants such frame work most. A query on the Thompson Web of Science database shows the importance of spreading in complex network, more 1500 papers for the year 2017. But the situation of eigensystem explanation for spreading problem do not go well. Valdano et al. reviewed past works and left a negative comment for applying eigenvector centralities method from static contacting network in year 2015 [5], they used statistics of contacting data.

This paper is organized as following. The following section shows the dynamic equation of spreading. Derivation of the dynamic equation is arranged in the appendix. The next following section shows the numerical result of non-time evolution network. The spreading speed of periodical repeating temporal network is after that. Future work is after the dissuasion section.

2 Derivation of the formula

For an individual-based simulation, one single network site indicates one agent only. We use binary value $\left(\overrightarrow{H}(t)\right)_i$ for describing state-of-been-spread for i-th agent at time t. While the value of $\left(\overrightarrow{H}(t)\right)_i$ is zero indicate the situation that i-th agent is free form been-spread state, and value one for been-spread state, respectively. The only condition of turning to be been-spread state of each agent is the existence of been-spread neighbour. Since a agent is in been-spread state, this been-spread agent will be remain in this state forever. We formulate this spreading dynamic of system with N agents on time a evoluting network:

$$\overrightarrow{H}(t) = \theta_1\left(\left(\overleftarrow{\prod_{t'=0}^{t-1}}\left(\widetilde{I} + \widetilde{A}(t')\right)\right)\overrightarrow{H}(0)\right) \tag{1}$$

$\widetilde{A}(t)$ is an adjacency matrix representation for temporal network at time t, $\widetilde{A}(t)_{i,j} = 1$ means that i-th and j-th agent is connected with a unidirectional link at time t, the value zero $\widetilde{A}(t)_{i,j} = 0$ means not linked, respectively. $\widetilde{I}$ is a $N \times N$ identity matrix. The symbol $\overleftarrow{\prod}$ is left-matrix-product notation, $\overleftarrow{\prod_{l=1}^{m}}\widetilde{A_l} = \widetilde{A_m}\ldots\widetilde{A_2}\widetilde{A_1}$. Where function $\theta_1(x)$ is a simplified denoting form unit step function, $\theta_1(x+1) = \theta(x)$. The appendix shows the properties of $\theta_1(x)$. The derivation of Eqn (1) is shown in the appendix with these $\theta_1(x)$'s properties.

3 Non-evolution matrix

The first step to revel the meaning of eigensystem is choosing the most simple case – non-evolution network and single spreading source, $\widetilde{A}(t) = \widetilde{A}$, $\left(\overrightarrow{H}(0)\right)_j = \delta_{i,j}$. The i-th agent is the unique spreading source. In the non-evolution network case, the Eqn (1) can be simplified as $\overrightarrow{H}(t) = \theta_1\left(\left(\widetilde{I}+\widetilde{A}\right)^t\overrightarrow{H}(0)\right)$, with employing Eqn (13). The been-spread state of j-th agent at time t will be $\theta_1\left(\left(\left(\widetilde{I}+\widetilde{A}\right)^t\right)_{ij}\right)$. This condition is the 100% been-spread starting from

i-th agent as a unique spreading origin:

$$\left(\left(\widetilde{I}+\widetilde{A}\right)^t\right)_{ij} \geqslant 1, \forall j. \tag{2}$$

Eigenmode decomposition is more easy comprehend for this matrix multiplication, $\left(\widetilde{I}+\widetilde{A}\right)^t_{ij} = \sum_k \left(\left(\lambda_k+1\right)^t w_{i,k} w_{j,k}\right)$. The k-th eigenpair of matrix $\widetilde{A}$ is λ_k and $\overrightarrow{W}_k$ is, such eigepairs satisfy the equation $\widetilde{A}\overrightarrow{W}_k = \lambda_k \overrightarrow{W}_k$. The i-th agent's eigenvector component in k-eigenvector is $w_{i,k}$, $w_{i,k} = \left(\overrightarrow{W}_1\right)_k$ The eigenpair indexes k are arranged as a descending order: $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_N$. While $k=1$, λ_1 and $w_{i,1}$ are called the principal eigenpair. The three similar matrices, $\left(\widetilde{I}+\widetilde{A}\right)^t$, $\widetilde{I}+\widetilde{A}$ and $\widetilde{A}$ shares the same eigenvector set. While condition

$$\left(\left(\lambda_1+1\right)/\left(\lambda_2+1\right)\right)^t \gg 1, \tag{3}$$

the principal eigenpair is suitable for estimate lower bound of S_i:

$$E^{\mathrm{lower}}\left(S_i\right) = -\frac{\log\left(w_{i,1} w_{j^{\min},1}\right)}{\log\left(\lambda_1+1\right)}. \tag{4}$$

Spanning tree If this non-time evolution network is spanning tree, the first agent, $i=1$, is the hub of this spanning tree, other agents link to the hub and there was no other links in this network. The off-diagonal elements of adjacency matrix is $\widetilde{A}_{ij} = \widetilde{A}_{ij} = 1$ if $i=1$. In this network, $S_1 = 1$ for $i=1$, and $S_i = 2$ for others. For understanding the asymptotic behaviour when system size goes to large $N \to \infty$, we denote a symbol $1/\alpha^2 = N-1$. The process of getting principal eigenpair of $\widetilde{A}$, λ_1 and $\overrightarrow{W}_1$, is following: to solve the eigenvector equation $\widetilde{A}\overrightarrow{W}_1 = \lambda_1\overrightarrow{W}_1$ with (a) positive eigenvector assumption $w_{i,1} = \left(\overrightarrow{W}_1\right)_i > 0$; (B) network symmetric $w_{j,1} = w_{l,1}$ $\forall j,l > 2$; (C) eigenvector normalization $\sum_i w_{i,1}^2 = 1$. The principal eigenpair are $\lambda_1 = 1/\alpha$,

$$w_{i,1} = \begin{cases} \frac{1}{\sqrt{\alpha^4+1}} & \text{if } i=1 \\ \frac{\alpha}{\sqrt{\alpha^4+1}} & \text{else} \end{cases}. \tag{5}$$

We can calculate the $E^{\mathrm{lower}}\left(S_i\right)$ of each agent from Eqn(4):

$$E^{\mathrm{lower}}\left(S_i\right) = \begin{cases} -\frac{\log\left(\frac{\alpha}{\alpha^4+1}\right)}{\log\left(\frac{1}{\alpha}+1\right)} \approx 1 + \frac{\alpha}{\log(\alpha)} + O\left(\alpha^2\right) & \text{if } i=1 \\ -\frac{2\log\left(\frac{\alpha}{\sqrt{\alpha^4+1}}\right)}{\log\left(\frac{1}{\alpha}+1\right)} \approx 2 + \frac{2\alpha}{\log(\alpha)} + O\left(\alpha^2\right) & \text{else} \end{cases}. \tag{6}$$

Such Taylor expansion shows the asymptotic behaviour when $N \to \infty$ as $\alpha \to 0$. The Figure (1) shows the numerical result of it.

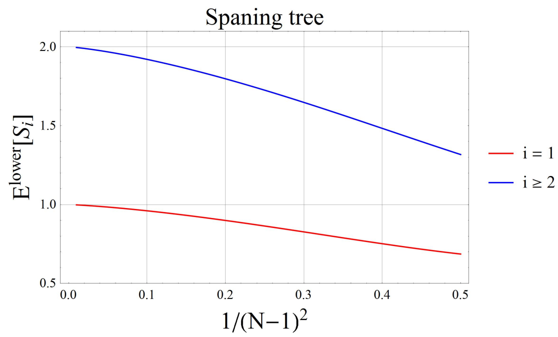

Fig. 1. Estimating step number to the end of each agent of spanning tree case. A one-level spanning tree is that hub agent, with the index $i = 1$, link all other agent $i = 2 \sim N$. In this network, $S_1 = 1$ for $i = 1$, and $S_i = 2$ for others. Eqn(4) is estimation formula that used. This numerical result show a asymptotic behaviour when system size goes to large The analytical form shows in Eqn(6).

Loop and it with and without an extra link We will show that our estimation formula remains the symmetric properties of the original network from comparing these two network. It is a $N = 16$ one dimensional loop, the degree (neighbours) of each agent is two. Such loop network contains symmetric properties and also can be found in the principal eigenvector components: $w_{i,1} = 1/\sqrt{N}$. The step number to coverage all the network should be half size of network, $S_i = N/2 = 8$. This result can be visualized as non zero matrix element figure of matrix $\left(\widetilde{I} + \widetilde{A}\right)^t$ in Fig (3). The value of estimation of S_i of each agent from principal eigenpair in Eqn (4) is the same :

$$E^{\mathrm{lower}}\left(S_i\right) = \frac{\log N}{\log 3} \approx 2.52372 \tag{7}$$

Where the principal eigenvalue is equal to average degree of network, $\lambda_1 = 2$. There no asymptotic behaviour when system size goes to infinity here, $\lim_{N\to\infty} E^{\mathrm{lower}}\left(S_i\right) \neq S_i$, because the $\lim_{N\to\infty} \lambda_1/\lambda_2 = \lim_{N\to\infty} 1/\left(\cos\left(2\pi/N\right)\right) = 1$. That do not fit the condition of $E^{\mathrm{lower}}\left(S_i\right)$ in Eqn (3). Such asymptotic eigenvalue degenercy will be break when we add a radius link. The radius link links first agent and $N/2 + 1$-th agent. The spectral properties can be calculate as a perturbation problem when system size goes to large. Form the fist step of perturbation calculation, the value of λ_1, $w_{1,1}$ and $w_{1,N/2+1}$ are larger then them in the simple loop, then the estimation values of S_i, S_1 and $S_{N/2+1}$ go to smaller. The calculation of eigensystem of this small system do no need perturbation method. The estimation value of S_i from principal eigenpair of this system is shown in Fig(3).

The S_i have three kind of sysmetric properties, our estimation also shows the same symmetric properties

$$S_1 = S_{n/2+1} \tag{8}$$

$$\begin{cases} S_{1+l} = S_{n-1-l} & l = 1, 2, \ldots, N/2 \\ S_{n/2+1+l} = S_{n/2+1+n-1} & l = 0, 1 \ldots, N/2 \end{cases}. \tag{9}$$

Combined the three symmetric properties, the system of N will partition as four group, within the edge node of group, the S_is have $N/4+1$ values. Therefore, the agents in the same set share the same value of S_i : {1,9}, {2,16,10,8} , {3,15,11,7} , {4,14,12,6}, {5,13}. The results of Fig (3) states that :$S_i = 3 + i$ $i = 1 \sim 5$, we can understand them from the following easy examples. In this network, the distance to the farest agent of 5-th and of 13 th agent is the same as them in the loop network without radius link $A_{1,N/2+l}$. The value of S_i for 5-th and of 13 th agent remains the same $S_5 = S_{13} = 8$. For the 4-th agent, the distance to the farest agent ,12-th, get one step smaller by shifting to the route with the radius link, the route $\{4 \to 3 \to 2 \to 1 \to 16 \to 15 \to 14 \to 13 \to 12\}$ to the route $\{4 \to 3 \to 2 \to 1 \to 9 \to 10 \to 11 \to 12\}$. These symmetric properties can also be found in our estimator because they are in the principal eigenvector, that can be revel by calculating higher order perturbations. These correspondence of symmetric is shown in Fig (3) as five $\{x, y\}$ points, otherwise it will shown more than five points. Comparing to the the relation of S_i and its lower bound estimator $E^{\text{lower}}(S_i)$, our estimator have the network symmetric properties and the monotonic relation to S_i.

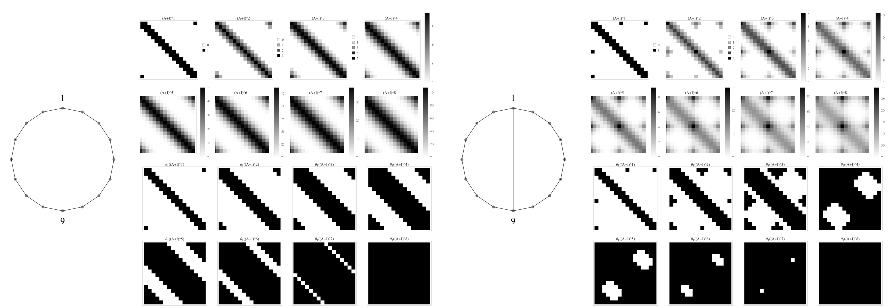

Fig. 2. Matrix multiplication of adjacency matrix. Using this multiplication matrix post mapped by the function $\theta_1(x)$, shown as black and white, the step number to the furthest agent can be got. In the left panel as a network without radius link, all columns or rows turn to black $\theta_1((\widetilde{A} + \widetilde{I})^t)_{ij} = 1$ while $t = 8$. Before that, at least on element in row is white. White matrix element $\theta_1((\widetilde{A} + \widetilde{I})^t)_{ij} = 0$ means that j-th agent can not be accessed by i-th agent. Therefore, $S_i = 8$ for all agent in the loop network without radius link. There are two white matrix element in the loop network with a radius link at $t = 7$. The two matrix elements are $\theta_1((\widetilde{A} + \widetilde{I})^7)_{5,13} = \theta_1((\widetilde{A} + \widetilde{I})^7)_{13,5} = 0$. It makes that$S_i = 8$ for $i = 13$ and 5. Other agents' S_i is shown in section 3. Pre mapped matrix by the function $\theta_1(x)$, shown as gray level, do not show S_i information clearly.

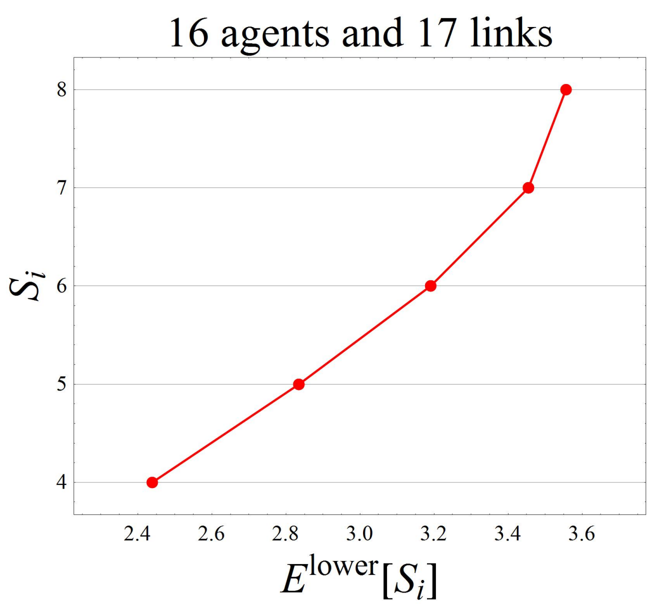

Fig. 3. Estimation value of S_i for the loop network with a radius link. The agent eigenvector component homogeneity in principal eigenpair among simple loop system is been break down by adding a radius link. But they remains the some systematic properties that S_i have. That the reason that why the estimation value of S_i in the system of $N = 16$ in this figure forms 5 points only. The value of S_i come from Fig (3). The estimation value of S_i come from Eqn (4). We discuss this figure in Section 3.

4 Temporal evolution network

For a system that the networks within $\widetilde{A}\,(t)$ repeat each τ step, $\widetilde{A}\,(t){=}\widetilde{A}\,(t+\tau)$, the time evolution of been-spread state $\overrightarrow{H}\,(t)$ in Eqn (1) can be denoted by a $\widetilde{P}$ matrix: $\overrightarrow{H}\,(t) = \theta_1\left(\left(\widetilde{P}\right)^{t/\tau}\overrightarrow{H}\,(0)\right)$, where

$$\widetilde{P} = \theta_1\left(\left(\overleftarrow{\prod_{t'=0}^{\tau-1}}\left(\widetilde{I}+\widetilde{A}\,(t')\right)\right).\right) \tag{10}$$

The following will show the $\widetilde{P}$ and its principal eigenair for two artificial case,$N = 3$ and $N = 60$, respectively. The results show that principal eigenair of $\widetilde{P}$ carries the dynamic information during the period τ.

$N = 3$, $\tau = 2$ case Each time step contains one single link of the route from first agent to the third agent via the second agent. The first step link the first and the second agent: $\left(\widetilde{A}\,(0)\right)_{1,2} = \left(\widetilde{A}\,(0)\right)_{2,1} = 1$, and $\left(\widetilde{A}\,(0)\right)_{i,j} = 0$ for else $\{i,j\}$ pair. The second step link the second and the third agent: $\left(\widetilde{A}\,(1)\right)_{2,3} = \left(\widetilde{A}\,(1)\right)_{3,2} = 1$, and $\left(\widetilde{A}\,(1)\right)_{i,j} = 0$ for else $\{i,j\}$ pair. The matrix expression of network $\widetilde{A}\,(t')$ is shown in the process of getting matrix $\widetilde{P}$

$$\widetilde{P} = \theta_1\left(\left(\widetilde{A}\,(1)+\widetilde{I}\right)\left(\widetilde{A}\,(0)+\widetilde{I}\right)\right) = \theta_1\left(\begin{pmatrix}1&0&0\\0&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}\right) \tag{11}$$

$$\rightarrow \widetilde{P} = \begin{pmatrix}1&1&0\\1&1&1\\1&1&1\end{pmatrix} \tag{12}$$

The three eigenvalues of matrix $\widetilde{P}$ in descending order are $\left\{\frac{1}{2}\left(3+\sqrt{5}\right),\frac{1}{2}\left(3-\sqrt{5}\right),0\right\}$. Their corresponding eigenvectors will be $\left\{\left\{\frac{1}{2}\left(\sqrt{5}-1\right),1,1\right\},\left\{\frac{1}{2}\left(-1-\sqrt{5}\right),1,1\right\},\{-1,1,0\}\right\}$, and equal to $\{\{0.618034, 1., 1.\}, \{-1.61803, 1., 1.\}, \{-1., 1., 0\}\}$ numerically. Comparing to the value of $w_{1,1}$, the larger eigenvector component in principal eigenpair of second and of third agent indicate they have higher risk of been-spread. This result can also tell from the $\widetilde{A}\,(t')$: the second agent and the third agent receives the spreading from other two agent, but the first agent can not receives the spreading from the third agent. It is fair that there are no degree of data compression during the process to get matrix $\widetilde{P}$ from $\widetilde{A}\,(t')$, six binary number pre and post the process. It is notable that a meaning of matrix elements of matrix $\widetilde{P}$'s transpose matrix $\widetilde{P}^{\mathrm{T}}$ is the possible-agent-spreading-from. Larger values of $\widetilde{P}^{\mathrm{T}}$'s principal eigenvector component shows more ability to spread things to other agent. That can be shown by this ration value in this system $w_{1,1}/w_{3,1} = w_{2,1}/w_{3,1} = 1.61$.

$N = 60$, $\tau = 5$ case. Permutation order of network in time will not change the average value in time, but it changes the result of spreading. This part shows a very simple example that the time order of network appearing changes the spreading importance of each agent. There are only two networks in this system,$\widetilde{A^{\#1}}$ and $\widetilde{A^{\#2}}$, and only one network appears in each time step. The two network and their relation can be understood as following procedure. First, loop the $N = 60$ agents as a ring. Partition this ring loop into $N_{\text{group}} = 10$ groups by taking N_{group} inter-group links. These N_{group} inter-group links form network #2, $\widetilde{A^{\#2}}$. The other $\left(N - N_{\text{group}}\right)$ intra-group links form network #1, $\widetilde{A^{\#1}}$. That results $\widetilde{A^{\#1}} + \widetilde{A^{\#2}} = \widetilde{A^{\text{ring}}}$. The edge of group is the i-th agent with this condition: $\mod\left(i, N/N_{\text{group}}\right) = 1$, or $\mod\left(i, N/N_{\text{group}}\right) = 0$. For example, first, 6-th and 60-th agent are the edge agent. We discuss two kind of permutation order in a $\tau = 5$ window, $\widetilde{A^{\#2}}$ first and $\widetilde{A^{\#2}}$ last. In first kind permutation order, $\widetilde{A^{\#2}}$ first, network $\widetilde{A^{\#2}}$ appears in the first step $\widetilde{A}(0) = \widetilde{A^{\#2}}$, and $\widetilde{A^{1}}$ appears in the following next four steps, $\widetilde{A}(t) = \widetilde{A^{\#1}}$ $t = 1 \sim 4$. This permutation order repeats every $\tau = 5$ steps: $\widetilde{A}(t) = \widetilde{A}(t-\tau)$. The beginning position of permutation order of network #2, $\widetilde{A}(0) = \widetilde{A^{\#2}}$, helps the edge agents spread to his neighbourhood group. Other non-edge can spread to his neighbourhood group since next τ repeating. The stronger spread ability to other agent of edge agents has been confirmed by the magnitude ratio principal eigenvector components shown in Fig (4) as blue points. In the second permutation order, the inter-group link was placed in the last $\widetilde{A}(4) = \widetilde{A^{\#2}}$, the ability of spreading of edge agents is suppressed. This result can also be understood from a perspective of principal eigenvector components in Fig (4) as orange points. In summary, the principal eigenvector components of matrix $\widetilde{P}$ or $\widetilde{P}^{\mathrm{T}}$ contain the information we need, the ability of spreading and the risk of been spread, respectively. This eigenvector representation is highly compressed. Post the normalization of eigenvector components, $\sim w_{i,1}^2 = 1$, we use $N - 1$ numbers to represent the information among $\tau N/2$ binary numbers.

5 Conclusion

Employing the sign function $\theta_1(x)$, the dynamic equation of spreading phenomena in a matrix multiplication expression in Eqn (1). That dynamic equation also states the importance of principal eigenpair. In a non-time evolution network system $\widetilde{A}(t) = \widetilde{A}$, principal eigenpair can estimate the step number to the furtherest agent S_i. In a time evolution network system $\widetilde{A}(t)$, the risk of been spread and the ability to spread is illustrated by the principal eigenvector of matrix $\widetilde{P}$ and of its transposed one $\widetilde{P}^{\mathrm{T}}$, respectively.

We find the asymptotic degeneracy for principal eigenvalues in Section 2. How the other eigenpair and degeneracy impact spreaing phenomena are arranged in our recent studies. We also will apply this method for studying the various

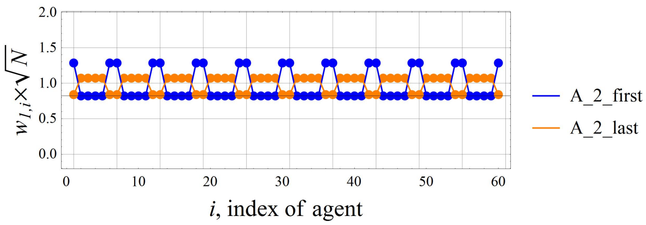

Fig. 4. Matrix $\widetilde{P}^{\mathrm{T}}$'s principal eigenvector component. The related magnitude of edge agents' principal eigenvector components to non-edge one rises while pushing the appearance time of inter-group-edge-network $\widetilde{A^{\#2}}$ in τ periodically repeat time window from the last position (orange color) to the first place (blue color). The first agent $i = 1$ is a typical example for edge agent, and $i = 2$ is non-edge one. That means that the related spreading ability increase for group-edge agents. The matrix $\widetilde{P}^{\mathrm{T}}$ is evaluating from the Eqn (10), notation $^{\mathrm{T}}$ for matrix transpose.

epidemic model besides traditional compartmental epidemic agent models [6] and for super-spreading phenomena and target vaccine problem.

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7 Appendex

7.1 Properties of function $\theta_1(x)$

Using the unit step function $\theta(d)$ to define a binary value function $\theta_1(d)$

$$\theta_1(d) = \theta(d-1) = \begin{cases} 1 & d \geqslant 1 \\ 0 & \texttt{else} \end{cases}$$

The following shows prosperities of function $\theta_1(d)$ for derivation of equations of spreading on network. In this system, the existence of link $\widetilde{A}_{ij}$ and spreading state of time $\left(\overrightarrow{H}(t)\right)_i$ are binary value. All the operations in this study are multiplication and addition without any subtraction. All values in this study should be non-negative integers. We shows prosperities of function $\theta_1(d)$ for non-negative integers. The symbols d and e are arbitrary non-negative integers , and $\overrightarrow{D}$, $\overrightarrow{E}$ are vector or matrix with arbitrary non-negative integers.

$$\left\{\begin{array}{l} \theta(d \times e - 1) = \theta(d-1) \times \theta(e-1) \\ \theta(d + e - 1) = \theta(\theta(d-1) + \theta(e-1) - 1) \end{array}\right\}$$

$$\rightarrow \left\{\begin{array}{l} \theta_1(de) = \theta_1(d)\,\theta_1(e) \\ \theta_1(d+e) = \theta_1(\theta_1(d) + \theta_1(e)) \end{array}\right\}$$

$$\rightarrow \theta_1(d_1e_1 + d_2e_2) = \theta_1(\theta_1(d_1e_1) + \theta_1(d_2e_2)) = \theta_1(\theta_1(d_1)\,\theta_1(e_1) + \theta_1(d_2)\,\theta_1(e_2))$$

We generalize the scalar function θ_1 to be a matrix function

$$\left(\theta_1\left(\overrightarrow{D}\right)\right)_{i,m} = \theta_1\left(\overrightarrow{D}_{i,m}\right)$$

$$\rightarrow \left\{\begin{array}{l} \theta_1\left(\overrightarrow{D}^T\overrightarrow{E}\right) = \theta_1\left(\theta_1\left(\overrightarrow{D}^T\right)\theta_1\left(\overrightarrow{E}\right)\right) \\ \theta_1\left(\overrightarrow{D} + \overrightarrow{E}\right) = \theta_1\left(\theta_1\left(\overrightarrow{D}\right) + \theta_1\left(\overrightarrow{E}\right)\right) \end{array}\right\}$$

The value of zero and one are the twos fix points of function $\theta_1(x) : \theta_1(0) = 0$, $\theta_1(1) = 1$. For a arbitrary binary matrix or vector $\overrightarrow{B}$, which is with element zero or one, $\overrightarrow{B}$ will be the same post been acted by θ_1:

$$\theta_1\left(\overrightarrow{B}\right) = \overrightarrow{B}. \tag{13}$$

And function value of $\theta_1(d)$ is binary, therefore for arbitrary matrix or vector

$$\theta_1\left(\theta_1\left(\overrightarrow{D}\right)\right) = \theta_1\left(\overrightarrow{D}\right).$$

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7.2 Detail derivation of Eqn 1

$$\overrightarrow{H}(t+1) = \overrightarrow{H}(t) + \theta_1 \left(\widetilde{A}(t)\,\overrightarrow{H}(t)\right)$$

$$\rightarrow \overrightarrow{H}(t+1) = \theta_1 \left(\overrightarrow{H}(t)\right) + \theta_1 \left(\widetilde{A}(t)\,\overrightarrow{H}(t)\right)$$

$$\rightarrow \overrightarrow{H}(t+1) = \theta_1 \left(\overrightarrow{H}(t) + \widetilde{A}(t)\,\overrightarrow{H}(t)\right)$$

$$= \theta_1 \left(\left(\widetilde{I} + \widetilde{A}(t)\right)\overrightarrow{H}(t)\right)$$

$$= \theta_1 \left(\theta_1 \left(\widetilde{I} + \widetilde{A}(t)\right)\overrightarrow{H}(t)\right)$$

$$\rightarrow \overrightarrow{H}(t+2) = \theta_1 \left(\theta_1 \left(\widetilde{I} + \widetilde{A}(t-1)\right)\theta_1 \left(\widetilde{I} + \widetilde{A}(t)\right)\overrightarrow{H}(t)\right)$$

$$\rightarrow \overrightarrow{H}(t) = \theta_1 \left(\left(\overleftarrow{\prod_{t'=0}^{t-1}}\theta_1 \left(\widetilde{I} + \widetilde{A}(t')\right)\right)\overrightarrow{H}(0)\right)$$

$$\rightarrow \overrightarrow{H}(t) = \theta_1 \left(\theta_1 \left(\overleftarrow{\prod_{t'=0}^{t-1}} \left(\widetilde{I} + \widetilde{A}(t')\right)\right)\overrightarrow{H}(0)\right)$$

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